

**2.1** On Minkowski spacetime  $(\mathbb{R}^{n+1}, \eta)$ , let  $(x^0, \dots, x^n)$  be the standard Cartesian coordinate system. Compute the induced metric on the submanifolds

$$S_{-1}^{(n,1)} = \left\{ - (x^0)^2 + \sum_{i=1}^n (x^i)^2 = -1 \right\}$$

and

$$S_{+1}^{(n,1)} = \left\{ - (x^0)^2 + \sum_{i=1}^n (x^i)^2 = +1 \right\}$$

(the latter is known as *de-Sitter* spacetime).

**Solution.** In order to compute the induced metric on those manifolds, we first have to find a convenient parametrization of them. For  $S_{-1}^{(n,1)}$ , it is convenient to express it as the graph of a function over the  $\{x^0 = 0\}$  hyperplane; that is to say, we define  $\Psi : \mathbb{R}^n \rightarrow S_{-1}^{(n,1)}$  by the relation:

$$\Psi(y^1, \dots, y^n) = (x^0, \dots, x^n) \quad \text{with} \quad x^\alpha = \begin{cases} \sqrt{1 + \|y\|^2}, & \text{if } \alpha = 0, \\ y^\alpha, & \text{if } \alpha \geq 1, \end{cases}$$

where

$$\|y\|^2 \doteq \sum_{i=1}^n (y^i)^2.$$

( $\Psi$  defined as above parametrizes only one of the two components of  $S_{-1}^{(n,1)}$ ; the other one is parametrized using  $x^0 = -\sqrt{1 + \|y\|^2}$  instead of  $x^0 = +\sqrt{1 + \|y\|^2}$ , but the resulting expression for the induced metric is the same). Then, in the coordinate chart  $\Psi^{-1}$  on  $S_{-1}^{(n,1)}$ , the induced metric  $g = \Psi^*\eta$  takes the form:

$$\begin{aligned} g = \Psi^*\eta &= -\Psi^*(dx^0)^2 + \sum_{i=1}^n \Psi^*(dx^i)^2 \\ &= -\left(d(\sqrt{1 + \|y\|^2})\right)^2 + \sum_{i=1}^n (dy^i)^2 \\ &= -\frac{\left(\sum_{i=1}^n y^i dy^i\right)^2}{1 + \|y\|^2} + \sum_{i=1}^n (dy^i)^2 \\ &= \left(\delta_{ij} - \frac{y^i y^j}{1 + \|y\|^2}\right) dy^i dy^j. \end{aligned}$$

Notice that the above metric is Riemannian, i.e. positive definite, since the matrix  $\delta_{ij} - \frac{y^i y^j}{1 + \|y\|^2}$  is positive definite (here, we used the fact that for any vector  $v \in \mathbb{R}^n$ , the matrix  $\mathbb{1} - v \otimes v$  has eigenvalues  $(1 - \|v\|^2, 1, \dots, 1)$ ). This is, of course, to be expected, since  $S_{-1}^{(n,1)}$  is a *spacelike* hypersurface of  $(\mathbb{R}^{n+1}, \eta)$ . In polar coordinates  $(r, \omega)$  on  $\mathbb{R}^n \setminus 0 \simeq (0, +\infty) \times \mathbb{S}^{n-1}$ ,  $g$  takes the form

$$g = \frac{1}{1 + r^2} dr^2 + r^2 g_{\mathbb{S}^{n-1}}(\omega).$$

The metric  $g$  is in fact isometric to the *hyperbolic* metric on  $\mathbb{R}^n$ .

Since  $S_+^{(n,1)}$  has the topology of a cylinder, we will use for it a parametrization by  $\mathbb{R} \times \mathbb{S}^{n-1}$ . To this end, let us switch to polar coordinates  $(x^0, r, \omega)$  on  $\mathbb{R}^{n+1}$  (so that  $r = \sqrt{\sum_{i=1}^n (x^i)^2} \in [0, +\infty)$  and  $\omega \in \mathbb{S}^{n-1}$  with  $\omega^i = \frac{x^i}{r}$ ,  $i \geq 1$ ). In these coordinates, the Minkowski metric  $\eta$  takes the form:

$$\eta = -(dx^0)^2 + dr^2 + r^2 g_{\mathbb{S}^{n-1}}(\omega),$$

while  $S_+^{(n,1)} = \left\{ (x^0, r, \omega) : -(x^0)^2 + r^2 = 1 \right\}$ . Let us consider the parametrization  $\Phi : \mathbb{R} \times \mathbb{S}^{n-1} \rightarrow S_+^{(n,1)}$  given by

$$\Phi(t, \theta) = (x^0, r, \theta) \quad \text{with} \quad x^0 = t, r = \sqrt{1+t^2}, \theta = \omega.$$

Then, the induced metric  $g_{dS}$  on  $S_+^{(n,1)}$  (the so-called *de-Sitter metric*) takes the following form

$$\begin{aligned} g_{dS} &= \Phi_* \eta = -\Phi_*(dx^0)^2 + \Phi_* dr^2 + r^2 \circ \Phi \cdot \Phi_* g_{\mathbb{S}^{n-1}}(\omega) \\ &= -dt^2 + \left( d(\sqrt{1+t^2}) \right)^2 + (1+t^2) g_{\mathbb{S}^{n-1}}(\theta) \\ &= -\frac{1}{1+t^2} dt^2 + (1+t^2) g_{\mathbb{S}^{n-1}}(\theta). \end{aligned}$$

**2.2** On Minkowski spacetime  $(\mathbb{R}^{n+1}, \eta)$ , let  $p, q \in \mathbb{R}^{n+1}$  be two points such that  $q \in I^+(p)$ . Let also  $\gamma_0 : [0, 1] \rightarrow \mathbb{R}^{n+1}$  be the straight line segment connecting  $p$  to  $q$  (i.e.  $\gamma_0(0) = p$ ,  $\gamma_0(1) = q$  and  $\ddot{\gamma}_0 = 0$ ) and  $\gamma : [0, 1] \rightarrow \mathbb{R}^{n+1}$  be any other *causal* curve such that  $\gamma(0) = p$  and  $\gamma(1) = q$ . Show that the corresponding lengths of the curves satisfy

$$\ell(\gamma_0) \geq \ell(\gamma).$$

This is a manifestation of the *twin paradox* in special relativity.

*Hint: Approximate  $\gamma$  by a polygonal causal curve and, using the inverse triangle inequality for causal vectors, show that the line segment connecting  $p$  and  $q$  has greater or equal length to a broken line segment connecting the same points.*

**Solution.** Let us first assume that the curve  $\gamma$  is a *polygonal*, future directed causal curve joining  $p$  and  $q$ , that is to say, there exist points  $\{p_k\}_{k=0}^N \in \mathbb{R}^{n+1}$  such that

1.  $p_0 = p$ ,  $p_N = q$ ,
2. The curve  $\gamma$  is the union of the line segments  $\overrightarrow{p_{k-1}p_k}$  connecting  $p_{k-1}$  to  $p_k$ ; explicitly, this means that there exists some partition  $\{t_k\}_{k=0}^N$  of  $[0, 1]$  such that  $t_0 = 0$ ,  $t_N = 1$  and

$$\gamma(s) = \frac{t_{k+1} - s}{t_{k+1} - t_k} p_k + \frac{s - t_k}{t_{k+1} - t_k} p_{k+1} \quad \text{if } s \in [t_k, t_{k+1}].$$

3. For all  $k = 0, \dots, N-1$ , the vectors  $\overrightarrow{p_{k-1}p_k}$  are *future directed* and causal.

Recall that the inverse triangle inequality states that, in any Lorentzian inner product space  $(V, m)$ , if  $v, w$  are two causal vectors belonging to the same component of the timecone, then

$$\|v + w\| \geq \|v\| + \|w\|.$$

In our case, noting that

$$\vec{pq} = \sum_{k=0}^{N-1} \vec{p_k p_{k+1}},$$

we have (thinking of those vectors as belonging to  $T_p \mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$  by translating them to have a starting point at  $p$ ):

$$\|\vec{pq}\|_{\eta_p} \geq \sum_{k=0}^{N-1} \|\vec{p_k p_{k+1}}\|_{\eta_p}.$$

Noting that, in this case

$$\ell(\gamma_0) = \|\vec{pq}\|_{\eta_p} \quad \text{and} \quad \ell(\gamma) = \sum_{k=0}^{N-1} \|\vec{p_k p_{k+1}}\|_{\eta_p},$$

we therefore have

$$\ell(\gamma_0) \geq \ell(\gamma).$$

Assume, now, that  $\gamma$  is a  $C^1$  causal curve; in that case, since  $\dot{\gamma}$  is continuous, it has to belong everywhere to the same connected component of the timecone, therefore it has to be future directed (since  $q \in I^+(p)$ ). We will show that, for any  $\epsilon > 0$  sufficiently small, there exists a point  $q_\epsilon \in I^+(p)$  and a *polygonal* future directed, causal curve  $\gamma_\epsilon : [0, 1] \rightarrow \mathbb{R}^{n+1}$  with  $\gamma_\epsilon(0) = p$ ,  $\gamma_\epsilon(1) = q_\epsilon$  and such that:

- $q_\epsilon \rightarrow q$  as  $\epsilon \rightarrow 0$ ,
- $|\ell(\gamma_\epsilon) - \ell(\gamma)| \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

Assume for a moment that such a point  $q_\epsilon$  and curve  $\gamma_\epsilon$  indeed exists for all sufficiently small  $\epsilon > 0$ . Then, since  $\gamma_\epsilon$  is a polygonal curve, by our previous argument we have

$$\|\vec{pq}_\epsilon\|_{\eta_p} \geq \ell(\gamma_\epsilon).$$

As  $\epsilon \rightarrow 0$ , we have (by our assumptions on  $q_\epsilon, \gamma_\epsilon$ ) that  $q_\epsilon \rightarrow q$  and  $\ell(\gamma_\epsilon) \rightarrow \ell(\gamma)$ . Therefore, since  $\|\vec{pq}\|_{\eta_p} = \ell(\gamma_0)$ , taking the limit  $\epsilon \rightarrow 0$  in the above inequality results in the desired statement:

$$\ell(\gamma_0) \geq \ell(\gamma).$$

It, therefore, remains to construct the approximating point  $q_\epsilon$  and polygonal curve  $\gamma_\epsilon$  satisfying Conditions 1 and 2 above. To this end, let us first note that any piecewise  $C^1$  curve  $\zeta : [0, 1] \rightarrow \mathbb{R}^{n+1}$

can be obtained from its derivative  $\dot{\zeta} : [0, 1] \rightarrow \mathbb{R}^{n+1}$  (here we view  $\dot{\zeta}(t)$  as a vector with base point translated to 0) by simply integrating componentwise:

$$\zeta(t) = \zeta(0) + \int_0^t \dot{\zeta}(s) ds$$

(in the above,  $+$  denotes simply summation in  $\mathbb{R}^{n+1}$ ) Thus, in order to approximate  $\gamma$  by a polygonal curve  $\gamma_\epsilon$ , we will first approximate  $\dot{\gamma}$  by a *piecewise constant* curve  $\tilde{\gamma}_\epsilon$  and then set

$$\gamma_\epsilon(t) = \gamma(0) + \int_0^t \tilde{\gamma}_\epsilon(s) ds.$$

Since  $\gamma$  is  $C^1$ ,  $\dot{\gamma}$  is continuous on  $[0, 1]$ ; since, in addition,  $[0, 1]$  is a compact interval,  $\dot{\gamma}$  must also be *uniformly* continuous. Therefore, for any  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon) > 0$  such that, for any  $t, s \in [0, 1]$  with  $|t - s| < \delta$  we have

$$\sum_{\alpha=0}^n |\dot{\gamma}^\alpha(t) - \dot{\gamma}^\alpha(s)| < \epsilon. \tag{1}$$

For any given  $\epsilon > 0$  and  $\delta = \delta(\epsilon)$  as above, let us consider the partition  $\{t_k\}_{k=0}^N$  of  $[0, 1]$  with  $N = \delta^{-1}$  such that

$$t_k = k\delta.$$

If we define, for  $k = 0, \dots, N - 1$ , the constant vectors

$$\dot{\gamma}_k \doteq \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \dot{\gamma}(t) dt,$$

the bound (1) implies that

$$\sup_{t \in [t_k, t_{k+1}]} \sum_{\alpha=0}^n |\dot{\gamma}^\alpha(t) - \dot{\gamma}_k^\alpha| < \epsilon. \tag{2}$$

Note also that, since  $\dot{\gamma}(t)$  is a causal, future directed vector and the future timecone is a convex set, the constants  $\dot{\gamma}_k$  are also causal and future directed.

Let  $v = (1, 0, \dots, 0)$ . We will define the piecewise constant curve  $\tilde{\gamma} : [0, 1) \rightarrow \mathbb{R}^{n+1}$  as follows:

$$\tilde{\gamma}_\epsilon(t) = \dot{\gamma}_k(t) + \epsilon v \quad \text{if } t \in [t_k, t_{k+1})$$

and we will set

$$\gamma_\epsilon(t) = \gamma(0) + \int_0^t \tilde{\gamma}_\epsilon(s) ds$$

and

$$q_\epsilon = \gamma_\epsilon(1).$$

Note the following facts:

- $\gamma_\epsilon$  is a polygonal curve connecting  $p$  to  $q_\epsilon$ ; the polygonal vertices of  $\gamma_\epsilon$  are the points  $p_k = \gamma_\epsilon(t_k)$  which can be explicitly computed from the formula of  $\gamma_\epsilon$  and the fact that  $\tilde{\gamma}_\epsilon$  is piecewise constant:

$$p_k = p + \sum_{l=0}^{k-1} \delta \cdot (\dot{\gamma}_l + \epsilon v).$$

- Since  $\dot{\gamma}_k$  is future directed causal and  $v$  is future directed timelike,  $\overrightarrow{p_k p_{k+1}} = \delta \cdot (\dot{\gamma}_k + \epsilon v)$  is future directed and timelike. Therefore,  $\overrightarrow{p q_\epsilon} = \sum_{k=0}^{N-1} \overrightarrow{p_k p_{k+1}}$  is also future directed and timelike (thus,  $q_\epsilon \in I^+(p)$ ).
- The definition of  $\dot{\gamma}_k$  implies that

$$\int_{t_k}^{t_{k+1}} \dot{\gamma}_k dt = \int_{t_k}^{t_{k+1}} \dot{\gamma}(t) dt.$$

Therefore, we have

$$\begin{aligned} q_\epsilon &= \gamma_\epsilon(1) \\ &= \gamma(0) + \int_0^1 \tilde{\gamma}_\epsilon(s) ds \\ &= \gamma(0) + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (\dot{\gamma}_k + \epsilon v) ds \\ &= \gamma(0) + \epsilon v + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \dot{\gamma}_k ds \\ &= \gamma(0) + \epsilon v + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \dot{\gamma}(s) ds \\ &= \gamma(0) + \epsilon v + \int_0^1 \dot{\gamma}(s) ds \\ &= \gamma(1) + \epsilon v \\ &= q + \epsilon v \end{aligned}$$

and, thus,  $q_\epsilon \rightarrow q$  as  $\epsilon \rightarrow 0$ .

- For any  $k = 0, \dots, N-1$  and any  $t \in (t_k, t_{k+1})$ , we have

$$\dot{\gamma}_\epsilon(t) = \dot{\gamma}_k + \epsilon v.$$

Thus, in view of the bound (2) on the difference between  $\dot{\gamma}(t)$  and  $\dot{\gamma}_k$ , we can estimate for any  $k = 0, \dots, N_1$  and any  $t \in (t_k, t_{k+1})$

$$\lim_{\epsilon \rightarrow 0} \left( \max_{\substack{k=0, \dots, N-1 \\ \alpha=0, \dots, n}} \sup_{t \in (t_k, t_{k+1})} |\dot{\gamma}^\alpha(t) - \dot{\gamma}_k^\alpha| + \epsilon \sum_{\alpha=0}^n |v^\alpha| \right) = 0$$

and, therefore (since  $\|\dot{\gamma}\|_\eta$  is a continuous function of  $\{\dot{\gamma}^\alpha\}_{\alpha=0}^n$ ):

$$\lim_{\epsilon \rightarrow 0} \left( \sup_{t \in (0,1)} \left| \|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta \right| \right) \rightarrow 0.$$

Therefore:

$$\begin{aligned} |\ell(\gamma) - \ell(\gamma_\epsilon)| &= \left| \sum_{k=0}^{N-1} \left( \int_{t_k}^{t_{k+1}} (\|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta) dt \right) \right| \\ &\leq \sum_{k=0}^{N-1} (t_{k+1} - t_k) \sup_{t \in (t_k, t_{k+1})} \left| \|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta \right| \\ &\leq \sup_{t \in (0,1)} \left| \|\dot{\gamma}_\epsilon(t)\|_\eta - \|\dot{\gamma}(t)\|_\eta \right| \\ &\xrightarrow{\epsilon \rightarrow 0} 0. \end{aligned}$$

The last two points above are precisely Conditions 1 and 2 on  $q_\epsilon$  and  $\gamma_\epsilon$ .

**2.3** Let  $(M, g)$  be a smooth Lorentzian *surface* (i.e. 2-dimensional manifold).

- (a) Show that for any  $p \in \mathcal{M}$ , there exists an open neighborhood  $\mathcal{U}$  of  $p$  and a local system of coordinates  $(u, v)$  on  $\mathcal{U}$  such that

$$g = \Omega(u, v) du dv,$$

where  $\Omega \in C^\infty(\mathcal{U})$  does not vanish in  $\mathcal{U}$  (such a coordinate system is called a *characteristic* or *double null* system).

- (b) Deduce that every smooth Lorentzian surface is locally conformally equivalent to an open subset of the Minkowski space  $(\mathbb{R}^{1+1}, \eta)$  (recall that a similar fact also holds for *Riemannian* surfaces; in that case, a coordinate system exhibiting this equivalence is called *isothermal*).

**Solution.** (a) For any  $p \in \mathcal{M}$ , let  $(x^1, x^2)$  be a local coordinate system in a neighborhood  $\mathcal{V}$  of  $p$ ; without loss of generality, we can assume that  $(x^1(p), x^2(p)) = (0, 0)$ . Since  $\mathcal{M}$  is two dimensional, the null cone  $N_q \subset T_q \mathcal{M}$  for every  $q \in \mathcal{V}$  consists of two intersecting straight lines  $\ell_q^{(1)}, \ell_q^{(2)} \subset T_q \mathcal{M}$ . Let us, therefore, choose two smooth *null* vector fields  $E_1$  and  $E_2$  on a (possibly smaller) open neighborhood  $\mathcal{V}'$  of  $p$  such that  $E_i|_q$  spans the null line  $\ell_q^{(i)}$  for every  $q \in \mathcal{V}'$ . Those vector fields can be constructed explicitly: If, for  $i = 1, 2$ ,  $E_i = E_i^1 \frac{\partial}{\partial x^1} + E_i^2 \frac{\partial}{\partial x^2}$  are the components of  $E_i$  in the  $(x^1, x^2)$  coordinate system, then the two vectors  $(E_i^1(q), E_i^2(q)) \in \mathbb{R}^2$  are roots of the quadratic form

$$Q_q(y^1, y^2) = g_{11}(q)(y^1)^2 + 2g_{12}(q)y^1 y^2 + g_{22}(q)(y^2)^2$$

on  $\mathbb{R}^2$ . This quadratic form has coefficients depending smoothly on  $q$  (since  $g$  was assumed to be smooth) and is of hyperbolic signature since  $g$  is Lorentzian (i.e.  $g_{12}^2 - g_{11}g_{22} < 0$ ); therefore, for

any  $q \in \mathcal{V}$ ,  $Q_q(y^1, y^2)$  the roots of  $Q_q(y^1, y^2)$  consist of two distinct straight lines of  $\mathbb{R}^2$  passing through 0 and depending smoothly on  $q$ . We can then choose  $(E_1^1(q), E_1^2(q))$  and  $(E_2^1(q), E_2^2(q))$  to be generators of these lines satisfying  $(E_i^1)^2(q) + (E_i^2)^2(q) = 1$ .<sup>1</sup>

We will construct the coordinate functions  $u, v : \mathcal{V}'' \subset \mathcal{V} \rightarrow \mathbb{R}$  by the requirement that their level sets in  $\mathcal{V}''$  (i.e. the curves  $\{u = \text{const}\}$  and  $\{v = \text{const}\}$ ) are *integral curves* of the vector fields  $E_1, E_2$ , respectively; note that this requirement does not uniquely determine  $u, v$ , since any reparametrization of the form  $u' = f_1 \circ u$  and  $v' = f_2 \circ v$  for smooth and invertible functions  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$  will have the same level sets.

To this end, let  $\gamma : \mathbb{R} \rightarrow \mathcal{M}$  be a smooth curve such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) \in T_p\mathcal{M} \setminus 0$  is transversal to both  $E_1|_p, E_2|_p$ . By continuity, there exists a  $\delta > 0$  such that  $\dot{\gamma}(s)$  is transversal to  $E_1|_{\gamma(s)}, E_2|_{\gamma(s)}$  for all  $s \in (-\delta, \delta)$ . Let us define the following open neighborhoods of  $p$ :

$$\mathcal{V}_1 = \{q \in \mathcal{V} : q \text{ belongs to an integral curve of } E_1 \text{ passing through } \gamma((-\delta, \delta))\}$$

and

$$\mathcal{V}_2 = \{q \in \mathcal{V} : q \text{ belongs to an integral curve of } E_2 \text{ passing through } \gamma((-\delta, \delta))\}$$

Let us also set  $\mathcal{V}'' = \mathcal{V}_1 \cap \mathcal{V}_2$ . Then, we can construct the functions  $u, v : \mathcal{V}'' \rightarrow \mathbb{R}$  by solving the following initial value problems with initial data on the curve  $\gamma((-\delta, \delta))$ :

$$\begin{cases} E_1(u) = 0, \\ u(\gamma(s)) = s \text{ for } s \in (-\delta, \delta) \end{cases} \quad \begin{cases} E_2(v) = 0, \\ v(\gamma(s)) = s \text{ for } s \in (-\delta, \delta). \end{cases}$$

Notice that  $(u(p), v(p)) = (0, 0)$  and  $u = v$  on the curve  $\gamma \cap \mathcal{V}'' = \gamma((-\delta, \delta))$ .

In order to say that  $(u, v)$  form a local system of coordinates around  $p$ , we have to show that the map  $(u, v) : \mathcal{V}'' \rightarrow \mathbb{R}^2$  is a diffeomorphism on its image in a neighborhood of  $p$ , or, equivalently, that the change of coordinates  $(x^1, x^2) \rightarrow (u, v)$  is non-singular in a neighborhood of  $(x^1(p), x^2(p)) = (0, 0)$ . By the inverse function theorem, it suffices to show that the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix}$$

is non-degenerate at  $p$ . To this end, we have to compute  $\partial_i u(p)$  and  $\partial_i v(p)$ . In view of the definition of the function  $u$ , at  $p = \gamma(0)$  we have

$$E_1(u)|_p = 0 \quad \text{and} \quad \dot{\gamma}(0)(u) = 1$$

which can be reexpressed in the  $(x^1, x^2)$  coordinate system as

$$\begin{cases} E_1^1 \frac{\partial u}{\partial x_1}(p) + E_1^2 \frac{\partial u}{\partial x_2}(p) = 0 \\ \dot{\gamma}^1(0) \frac{\partial u}{\partial x_1}(p) + \dot{\gamma}^2(0) \frac{\partial u}{\partial x_2}(p) = 1 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} E_1^1(p) & E_1^2(p) \\ \dot{\gamma}^1(0) & \dot{\gamma}^2(0) \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x_1}(p) \\ \frac{\partial u}{\partial x_2}(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

<sup>1</sup>A different way of choosing  $(E_i^1(q), E_i^2(q))$  could be as follows: Assuming without loss of generality that  $g_{11}(p) \neq 0$  (and hence  $g_{11} \neq 0$  in a neighborhood of  $p$ ), then, setting  $\lambda_i = \frac{E_i^1}{E_i^2}$ , we obtain that  $\lambda_i$  must satisfy the quadratic equation  $g_{11}(q)\lambda_i^2 + 2g_{12}(q)\lambda_i + g_{22} = 0$ , so we can pick  $\lambda_1(q)$  and  $\lambda_2(q)$  to be the two roots of this equation.

Since we assumed that the vectors  $\dot{\gamma}(0), E_1 \in T_p\mathcal{M}$  are transversal (i.e. not collinear), the matrix on the left hand side above has rank 2 and is therefore invertible; we can thus write:

$$\begin{bmatrix} \frac{\partial u}{\partial x_1}(p) \\ \frac{\partial u}{\partial x_2}(p) \end{bmatrix} = \begin{bmatrix} E_1^1(p) & E_1^2(p) \\ \dot{\gamma}^1(0) & \dot{\gamma}^2(0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Working similarly for the function  $v$ , we infer:

$$\begin{bmatrix} \frac{\partial v}{\partial x_1}(p) \\ \frac{\partial v}{\partial x_2}(p) \end{bmatrix} = \begin{bmatrix} E_2^1(p) & E_2^2(p) \\ \dot{\gamma}^1(0) & \dot{\gamma}^2(0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since the vectors  $E_1|_p, E_2|_p$  are not collinear, we can readily compute that the vectors  $\begin{bmatrix} \frac{\partial u}{\partial x_1}(p) \\ \frac{\partial u}{\partial x_2}(p) \end{bmatrix}$  and  $\begin{bmatrix} \frac{\partial v}{\partial x_1}(p) \\ \frac{\partial v}{\partial x_2}(p) \end{bmatrix}$  obtained from the above expressions (using the formula for the inverse of a  $2 \times 2$  matrix) are not collinear either; thus, the Jacobian matrix  $J(p)$  has rank 2 and is, therefore, invertible. Hence,  $(u, v)$  define a coordinate chart in a neighborhood of  $p$ .

In the  $(u, v)$  coordinate system, the coordinate vector field  $\frac{\partial}{\partial u}$  is tangent to the coordinate curve  $\{v = \text{const}\}$ . In view of the fact that  $E_2(v) = 0$ , this means that  $\frac{\partial}{\partial u} \parallel E_2$  and, therefore,  $\frac{\partial}{\partial u}$  is *null*, i.e.

$$g_{uu} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = 0.$$

Similarly,  $\frac{\partial}{\partial v} \parallel E_1$  and, therefore,

$$g_{vv} = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = 0.$$

As a result,

$$\begin{aligned} g &= g_{uu}(du)^2 + 2g_{uv}dudv + g_{vv}(dv)^2 \\ &= 2g_{uv}dudv. \end{aligned}$$

Of course,  $g_{uv}$  cannot vanish at any point in the region where  $(u, v)$  is a coordinate system (if  $g_{uv}(q) = 0$ , then  $g|_q = 0$ , which would violate the assumption that  $g|_q$  is non-degenerate). Therefore,  $g_{uv}$  has a constant sign; by switching  $u \rightarrow -u$  if necessary, we can assume that  $g_{uv} > 0$ . Thus, setting

$$\Omega \doteq 2g_{uv},$$

we obtain the required expression.

(b) On Minkowski spacetime  $(\mathbb{R}^{1+1}, \eta)$ , we can introduce the standard double null coordinate system by setting

$$\begin{aligned} \bar{v} &= x^0 + x^1, \\ \bar{u} &= x^0 - x^1. \end{aligned}$$

In these coordinates, the Minkowski metric  $\eta$  takes the form

$$\begin{aligned} \eta &= -(dx^0)^2 + (dx^1)^2 \\ &= -\left(\frac{d\bar{v} + d\bar{u}}{2}\right)^2 + \left(d\frac{\bar{v} - d\bar{u}}{2}\right)^2 \\ &= -d\bar{u}d\bar{v}. \end{aligned}$$

Returning to our 2-dimensional Lorentzian manifold  $(\mathcal{M}, g)$ , for any point  $p \in \mathcal{M}$ , let  $(u, v)$  be the double null coordinate system in a neighborhood  $\mathcal{V}$  of  $p$  constructed in part (a); recall that, in these coordinates

$$g = \Omega dudv.$$

Let us define the map  $\phi : \mathcal{V} \rightarrow \mathbb{R}^{1+1}$  so that, in the  $(\bar{u}, \bar{v})$  coordinates on  $\mathbb{R}^{1+1}$ :

$$(\phi^{\bar{u}}(u, v), \phi^{\bar{v}}(u, v)) = (u, v)$$

Then we can immediately compute that

$$\begin{aligned} \phi_*\eta &= (\phi_*d\bar{u})(\phi_*d\bar{v}) \\ &\stackrel{\substack{\bar{u}(u,v)=u \\ \bar{v}(u,v)=v}}{=}}{=} dudv \end{aligned}$$

and, therefore,

$$\phi_*\eta = \Omega^{-1} \cdot g.$$

Therefore, the map  $\phi : (\mathcal{V}, g) \rightarrow (\phi(\mathcal{V}), \eta)$  is conformal.

**\*2.4** In this exercise, we will show that there are topological obstructions to a manifold admitting a Lorentzian metric; **not** every smooth manifold admits one. To this end, let us adopt the following definition: For any Lorentzian inner product space  $(V, m)$ , we will call any 2-element set of the form  $\{u, -u\}$  (where  $u \in V \setminus 0$ ) a *line seed*. A line seed  $X = \{u, -u\}$  will be called *causal* if  $u \in V$  is a causal vector. We will also define the *trivial* line seed to be the pair  $\{0, -0\}$ .

Given two causal line seeds  $X = \{u, -u\}$  and  $Y = \{v, -v\}$ , then exactly one of the vectors  $+v$  and  $-v$  belongs to the same timecone as  $u$ . We will define the sum  $X + Y$  as the seed  $\{u + v, -u - v\}$  if  $u, v$  belong to the same time cone and as  $\{u - v, -u + v\}$  otherwise. We will extend this definition to include the trivial line seed.

- (a) Verify that, with the addition operator defined above,  $X_1 + X_2$  is a causal line seed if  $X_1, X_2$  are causal line seeds or if one of them is causal and the other is the trivial line seed.
- (b) Let  $(\mathcal{M}, g)$  be a smooth Lorentzian manifold and let  $p \in \mathcal{M}$ . Show that there exists an open neighborhood  $\mathcal{U}$  of  $p$  and a smooth causal vector field  $U \in \Gamma(\mathcal{U})$ .
- (c) A smooth *line field seed* on  $\mathcal{M}$  will be an assignment of a line seed  $X_p = \{U_p, -U_p\}$  in  $T_p\mathcal{M}$  for each  $p \in \mathcal{M}$  such that, for any  $q \in \mathcal{M}$ , there exists an open neighborhood  $\mathcal{V}$  of

$q$  and a smooth vector field  $Y \in \Gamma(\mathcal{V})$  such that  $Y(p) \in X_p$  for all  $p \in \mathcal{V}$ .<sup>2</sup> Show that  $\mathcal{M}$  as above admits a smooth *causal* line field seed.

*Hint: For this part, it might be helpful to use the fact that any smooth manifold admits a **partition of unity**: For any open covering  $\{\mathcal{U}_a\}_a$  of  $\mathcal{M}$ , there exists a family  $\{\chi_\beta\}_\beta$  of smooth functions  $\chi_\beta : \mathcal{M} \rightarrow [0, +\infty)$  satisfying the following properties:*

- \* Each  $\chi_\beta$  is compactly supported, and its support is contained in one of the open sets  $\mathcal{U}_a$ .
- \* For each  $\chi_\beta$ ,  $\text{supp}(\chi_\beta)$  intersects only finitely many of the supports of  $\chi_\gamma$ ,  $\gamma \neq \beta$ .
- \* For any  $p \in \mathcal{M}$ ,  $\sum_\beta \chi_\beta(p) = 1$ .

You can then use part 2.4.b to construct a smooth causal line seed field in a neighborhood of every point in  $\mathcal{M}$ , and then use an appropriate partition of unity to “glue” these constructions together, utilising the notion of the sum of two causal line seeds from part 2.4.a.

- (d) Deduce that the tangent bundle  $T\mathcal{M}$  of  $\mathcal{M}$  admits a smooth line subbundle. Can the sphere  $S^2$  admit a Lorentzian metric?

*Hint: Use the fact that, for a compact manifold  $\mathcal{M}$ , if the tangent bundle admits a line subbundle then the Euler characteristic  $\chi(\mathcal{M})$  of  $\mathcal{M}$  vanishes.*

**Solution.** (a) It is easy to verify using the fact that each connected component  $C_+$  and  $C_-$  of the causal cone  $C = \{v \in V \setminus 0 : m(v, v) \leq 0\}$  is a convex cone that if  $u, v$  are causal vectors, then  $u + v$  is also a causal vector in the same cone. From this, it readily follows that  $X + Y$ , as defined in the statement of the exercise, is a causal line seed if  $X$  and  $Y$  are causal line seeds or if one of them is causal and the other the trivial line seed.

For the rest of this exercise, we will denote with  $\mathcal{F}(V)$  the set of line seeds in  $V$  which are either causal or trivial. Note that  $+$  is well defined on  $\mathcal{F}(V) \times \mathcal{F}(V)$  and is associative, commutative and has a unique zero element (the trivial line seed).

(b) Let  $(x^0, \dots, x^n)$  be a local system of coordinates in a neighborhood  $\mathcal{V}$  of  $p$ . Let  $U_p \in T_p\mathcal{M} \setminus 0$  be a timelike vector, with corresponding components  $\{U_p^\alpha\}_{\alpha=0}^n$ . We can then define the smooth vector field  $U$  on  $\mathcal{V}$  by the relation

$$U = U_p^\alpha \frac{\partial}{\partial x^\alpha},$$

i.e. the components of  $U$  in the  $(x^0, \dots, x^n)$  coordinate system are constant functions ( $U^\alpha = U_p^\alpha$ ). Since the metric  $g$  is smooth, the set of timelike vectors, i.e. the set

$$\mathcal{I} = \{(q, v) \in T\mathcal{M} : g_q(v, v) < 0\}$$

is an open subset of the tangent bundle  $T\mathcal{M}$ . Therefore, since  $U|_p$  was assumed to be timelike vector in  $T_p\mathcal{M}$ ,  $U|_q$  will also be timelike (and hence causal) for any point  $q$  in a sufficiently small open neighborhood  $\mathcal{U}$  of  $p$ .

<sup>2</sup>Note that, with this definition, the tangent vector  $U_p$  need not even be continuous in  $p$ ; we only require that there is (locally at least) a choice between  $U_p$  and  $-U_p$  at every point  $p$  that results in a smooth vector field.

(c) From part (b), we know that, for every point  $p \in \mathcal{M}$ , there exists an open neighborhood  $\mathcal{U}^{(p)}$  of  $p$  and a smooth causal vector field  $T^{(p)}$  defined on  $\mathcal{U}^{(p)}$ . We can, therefore, also define the causal line field seed  $X^{(p)}$  on  $\mathcal{U}^{(p)}$  by the relation

$$X^{(p)}|_q = \{T^{(p)}|_q, -T^{(p)}|_q\} \text{ for any } q \in \mathcal{U}^{(p)}.$$

The collection of open sets  $\{\mathcal{U}^{(p)}\}_{p \in \mathcal{M}}$  covers the whole of  $\mathcal{M}$  (since, for any  $q \in \mathcal{M}$ ,  $q \in \mathcal{U}^{(q)}$ ). We can therefore introduce a partition of unity  $\{\chi_\beta\}_\beta$  subordinate to the open cover  $\{\mathcal{U}^{(p)}\}_{p \in \mathcal{M}}$ ; this is a set of smooth functions  $\chi_\beta : \mathcal{M} \rightarrow [0, +\infty)$  (where the set of indices  $\beta$  is not necessarily the same as the index set for the open cover) satisfying the following properties:

1. For any  $\beta$ , there exists a  $p = p(\beta)$  such that  $\text{supp}(\chi_\beta) \subset \mathcal{U}^{(p)}$ .
2. For any  $p \in \mathcal{M}$ , there exists an open neighborhood  $\mathcal{V}_p$  of  $p$  such that only finitely many of the functions  $\chi_\beta$  are supported on  $\mathcal{V}_p$ .
3. For any  $p \in \mathcal{M}$ , we have

$$\sum_{\beta} \chi_\beta(p) = 1$$

(by the previous property, this is a finite sum).

The fact that such a partition of unity always exists for any open cover of a smooth manifold is a fundamental result in the theory of manifolds; see for example the book by Brickel and Clark: *Differentiable manifolds: An Introduction*.

For a partition of unity  $\{\chi_\beta\}_\beta$  as above, let us define a (non-unique) map  $\beta \rightarrow p(\beta) \in \mathcal{M}$  so that  $\text{supp}\chi_\beta \subset \mathcal{U}^{(p)}$  (such a  $p$  exists by property 1 above). For any  $\beta$ , let us consider the pair of vector fields defined on  $\mathcal{U}^{(p(\beta))}$

$$X_\beta = \{\chi_\beta \cdot T^{(p(\beta))}, -\chi_\beta \cdot T^{(p(\beta))}\}.$$

Since the support of  $\chi_\beta$  is contained in  $\mathcal{U}^{(p(\beta))}$ , the vector fields  $\pm\chi_\beta \cdot T^{(p(\beta))}$  can be smoothly extended on the whole of  $\mathcal{M}$  by simply assuming that they vanish identically on  $\mathcal{M} \setminus \mathcal{U}^{(p(\beta))}$ . In this way, the pair  $X_\beta$  is now a pair of smooth vector fields on the whole of  $\mathcal{M}$ ; at any point  $q \in \mathcal{M}$ ,  $X_\beta|_q \in \mathcal{F}(T_q\mathcal{M})$  (see the end of the solution of part (a) for the notation  $\mathcal{F}(V)$ ) and, moreover, at any point  $q$  such that  $\chi_\beta(q) > 0$ , the pair  $X_\beta|_q$  is a *causal line seed* (note that  $X_\beta$  is the trivial line seed on  $\overline{\mathcal{M} \setminus \text{supp}\chi_\beta}$ ).

Let us consider, now, for any point  $q \in \mathcal{M}$ , the following element of  $\mathcal{F}(T_q\mathcal{M})$  (see, again, the end of the solution of part (a) for the and the addition operation on  $\mathcal{F}(V)$ ):

$$X|_q = \sum_{\beta} X_\beta|_q.$$

This is a finite sum, since only a finite number of the  $X_\beta$ 's are non-zero at  $q$ ; moreover, the sum is well defined as an element of  $\mathcal{F}(T_q\mathcal{M})$  because, for all  $\beta$ ,  $X_\beta|_q \in \mathcal{F}(T_q\mathcal{M})$ . Moreover, the following properties hold:

- For any point  $p \in \mathcal{M}$ , using the timelike vector field  $T^{(p)}$  defined on  $\mathcal{U}^{(p)}$ , we can define a continuous assignment of a future directed causal cone to the tangent spaces  $T_q\mathcal{M}$  for all  $q \in \mathcal{U}^{(p)}$ : For any  $q \in \mathcal{U}^{(p)}$  the *future directed* component  $C_q^+$  of the causal cone  $C_q = \{v \in T_q\mathcal{M} : v \text{ is causal}\}$  can be fixed by the condition:

$$C_q^+ = \{v \in C_q : g(v, T^{(p)}) < 0\}.$$

(since  $T^{(p)}$  is a smooth vector field, this assignment of a future directed component is continuous in  $q \in \mathcal{W}_p$ ). Therefore, for any one of the vector field pairs  $X_\beta = \{\chi_\beta \cdot T^{(p(\beta))}, -\chi_\beta \cdot T^{(p(\beta))}\}$  restricted over  $\text{supp}\chi_{\beta(p)}$ , we can distinguish one future directed and one past directed element of  $X_\beta|_{\mathcal{U}^{(p)}}$ ; we will denote with  $X_\beta|_{\mathcal{U}^{(p)}}^+$  the future directed vector field among the pair  $\{\chi_\beta \cdot T^{(p(\beta))}|_{\mathcal{U}^{(p)}}, -\chi_\beta \cdot T^{(p(\beta))}|_{\mathcal{U}^{(p)}}\}$  and with  $X_\beta|_{\mathcal{U}^{(p)}}^-$  the past directed one. We will extend this definition outside  $\text{supp}\chi_{\beta(p)}$  trivially, since  $X_\beta = 0$  there. Thus,

$$X_\beta|_{\mathcal{U}^{(p)}} = \{X_\beta|_{\mathcal{U}^{(p)}}^+, X_\beta|_{\mathcal{U}^{(p)}}^-\}.$$

Having this distinction between future and past directed causal vector fields over  $\mathcal{U}^{(p)}$ , it is then easy to see that  $X$  restricted to  $\mathcal{U}^{(p)}$  can be written as

$$\begin{aligned} X|_{\mathcal{U}^{(p)}} &= \{X|_{\mathcal{U}^{(p)}}^+, X|_{\mathcal{U}^{(p)}}^-\} \\ &= \left\{ \sum_\beta X_\beta|_{\mathcal{U}^{(p)}}^+, \sum_\beta X_\beta|_{\mathcal{U}^{(p)}}^- \right\}. \end{aligned}$$

In particular, the line seed  $X|_{\mathcal{U}^{(p)}}$  can be written as a pair of smooth vector fields; therefore  $X$  is a *smooth* line field.

- For any point  $p \in \mathcal{M}$ , there exists at least one  $\beta$  such that  $\chi_\beta(p) > 0$  (since  $\sum_\beta \chi_\beta(p) = 1$ ). Therefore, not all vector fields  $X_\beta|_{\mathcal{U}^{(p)}}^+$  vanish at  $p$ ; as a result,  $X|_p \in \mathcal{F}(T_p\mathcal{M})$  is not the pair  $\{0, 0\}$  and is therefore a *causal* line seed.

Thus, we have shown that  $X$  is a *smooth, causal* line seed field on  $\mathcal{M}$ .

(d) The existence of a smooth line seed field  $X$  on  $\mathcal{M}$  determines a line subbundle of  $T\mathcal{M}$ , namely the subbundle  $E \hookrightarrow T\mathcal{M}$  spanned by the two elements of  $X|_p = \{x_p, -x_p\}$  at each point  $p \in \mathcal{M}$ :

$$E = \{(p, v) \in T\mathcal{M} : v = \lambda x_p \text{ for some } \lambda \in \mathbb{R} \text{ and } x_p \in X|_p\}.$$

It is known from algebraic topology that, if the tangent bundle  $T\mathcal{M}$  of a smooth compact manifold  $\mathcal{M}$  admits a smooth line subbundle, then the *Euler characteristic* of the manifold  $\mathcal{M}$  (which can be computed as the alternating sum  $\sum_{k=0}^n (-1)^k F_k$  of the  $k$ -dimensional faces in a finite triangulation of the manifold  $\mathcal{M}$ ). The Euler characteristic of the sphere  $\mathbb{S}^n$  is  $1 + (-1)^n$ ; thus, for  $n \in 2\mathbb{Z}$ , the tangent bundle of the sphere  $\mathbb{S}^n$  cannot admit a smooth line subbundle and, therefore,  $\mathbb{S}^n$  in this case cannot admit a smooth Lorentzian metric.

**Bonus exercise (hard):** Can you construct a Lorentzian metric on  $\mathbb{S}^3$ ? (*Hint: Use the Hopf fibration  $\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2$  to foliate  $\mathbb{S}^3$  by 1-dimensional circles and use a vector field tangent to those circles to construct suitable timecones.*)